

Fundamentals of Linear Algebra and Optimization
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Homework 5 & 6 + Project 3 & 4

Note: Problems B2 and B6 are for extra credit

April 17, 2012; Due May 7, 2012

Problem B1 (20 pts). Let A be any real or complex $n \times n$ matrix and let $\| \cdot \|$ be any operator norm.

Prove that for every $m \geq 1$,

$$\|I\| + \sum_{k=1}^m \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$

Deduce from the above that the sequence (E_m) of matrices

$$E_m = I + \sum_{k=1}^m \frac{A^k}{k!}$$

converges to a limit denoted e^A , and called the *exponential* of A .

Problem B2 (Extra Credit 60 pts). Recall that the affine maps $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where $\theta, w_1, w_2 \in \mathbb{R}$, are *rigid motions* (or *direct affine isometries*) and that they form the group **SE**(2).

Given any map ρ as above, if we let

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then ρ can be represented by the 3×3 matrix

$$A = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & w_1 \\ \sin \theta & \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

(a) Consider the set of matrices of the form

$$\begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^3, +)$. This vector space is denoted by $\mathfrak{se}(2)$. Show that in general, $BC \neq CB$, if $B, C \in \mathfrak{se}(2)$.

(b) Given a matrix

$$B = \begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix},$$

prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} \cos \theta & -\sin \theta & \frac{u}{\theta} \sin \theta + \frac{v}{\theta} (\cos \theta - 1) \\ \sin \theta & \cos \theta & \frac{u}{\theta} (-\cos \theta + 1) + \frac{v}{\theta} \sin \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

Hint. Prove that

$$B^3 = -\theta^2 B,$$

and that

$$e^B = I_3 + \frac{\sin \theta}{\theta} B + \frac{1 - \cos \theta}{\theta^2} B^2.$$

(c) Check that e^B is a direct affine isometry in $\mathbf{SE}(2)$. Prove that the exponential map $\exp: \mathfrak{se}(2) \rightarrow \mathbf{SE}(2)$ is surjective. If $\theta \neq k2\pi$ ($k \in \mathbb{Z}$), how do you need to restrict θ to get an injective map?

Problem B3 (100 + 200 pts). Consider the affine maps $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where $\theta, w_1, w_2, \alpha \in \mathbb{R}$, with $\alpha > 0$. These maps are called (direct) *affine similitudes* (for short, *similitudes*). The number $\alpha > 0$ is the *scale factor* of the similitude. These affine maps are the composition of a rotation of angle θ , a rescaling by $\alpha > 0$, and a translation.

(a) Prove that these maps form a group that we denote by **SIM**(2).

Given any map ρ as above, if we let

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then ρ can be represented by the 3×3 matrix

$$A = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha \cos \theta & -\alpha \sin \theta & w_1 \\ \alpha \sin \theta & \alpha \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha R X + W.$$

(b) Consider the set of matrices of the form

$$\begin{pmatrix} \lambda & -\theta & u \\ \theta & \lambda & v \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta, \lambda, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^4, +)$. This vector space is denoted by **sim**(2).

(c) Given a matrix

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

prove that

$$e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Hint. Write

$$\Omega = \lambda I + \theta J,$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $J^2 = -I$, and prove by induction on k that

$$\Omega^k = \frac{1}{2} ((\lambda + i\theta)^k + (\lambda - i\theta)^k) I + \frac{1}{2i} ((\lambda + i\theta)^k - (\lambda - i\theta)^k) J.$$

(d) As in (c), write

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

let

$$U = \begin{pmatrix} u \\ v \end{pmatrix},$$

and let

$$B = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}.$$

Prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}U \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_2$.

Prove that

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(e) Prove that

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!} = \int_0^1 e^{\Omega t} dt.$$

Use this formula to prove that if $\lambda = \theta = 0$, then

$$V = I_2,$$

else

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix}.$$

Observe that for $\lambda = 0$, the above gives back the expression in B2(b) for $\theta \neq 0$.

Conclude that if $\lambda = \theta = 0$, then

$$e^B = \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix},$$

else

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

with

$$e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

and

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix},$$

and that $e^B \in \mathbf{SIM}(2)$, with scale factor e^λ .

(f) Prove that the exponential map $\exp: \mathfrak{sim}(2) \rightarrow \mathbf{SIM}(2)$ is surjective.

(g) Similitudes can be used to describe certain deformations (or flows) of a deformable body \mathcal{B}_t in the plane. Given some initial shape \mathcal{B} in the plane (for example, a circle), a deformation of \mathcal{B} is given by a piecewise differentiable curve

$$\mathcal{D}: [0, T] \rightarrow \mathbf{SIM}(2),$$

where each $\mathcal{D}(t)$ is a similitude (for some $T > 0$). The deformed body \mathcal{B}_t at time t is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}).$$

The surjectivity of the exponential map $\exp: \mathfrak{sim}(2) \rightarrow \mathbf{SIM}(2)$ implies that there is a map $\log: \mathbf{SIM}(2) \rightarrow \mathfrak{sim}(2)$, although it is multivalued. The exponential map and the log “function” allows us to work in the simpler (noncurved) Euclidean space $\mathfrak{sim}(2)$.

For instance, given two similitudes $A_1, A_2 \in \mathbf{SIM}(2)$ specifying the shape of \mathcal{B} at two different times, we can compute $\log(A_1)$ and $\log(A_2)$, which are just elements of the Euclidean space $\mathfrak{sim}(2)$, form the linear interpolant $(1 - t)\log(A_1) + t\log(A_2)$, and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1) + t\log(A_2)}, \quad t \in [0, 1].$$

Also, given a sequence of “snapshots” of the deformable body \mathcal{B} , say A_0, A_1, \dots, A_m , where each is A_i is a similitude, we can try to find an interpolating deformation (a curve in $\mathbf{SIM}(2)$) by finding a simpler curve $t \mapsto C(t)$ in $\mathfrak{sim}(2)$ (say, a B -spline) interpolating $\log A_0, \log A_1, \dots, \log A_m$. Then, the curve $t \mapsto e^{C(t)}$ yields a deformation in $\mathbf{SIM}(2)$ interpolating A_0, A_1, \dots, A_m .

(1) **(75 pts)**. Write a program interpolating between two deformations.

(2) **(125 pts)**. Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes A_0, A_1, \dots, A_m .

Problem B4 (40 pts). Recall that the coordinates of the cross product $u \times v$ of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in \mathbb{R}^3 are

$$(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

(a) If we let U be the matrix

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

check that the coordinates of the cross product $u \times v$ are given by

$$\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

(b) Show that the set of matrices of the form

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

is a vector space isomorphic to $(\mathbb{R}^3, +)$. This vector space is denoted by $\mathfrak{so}(3)$. Show that such matrices are never invertible. Find the kernel of the linear map associated with a matrix U . Describe geometrically the action of the linear map defined by a matrix U . Show that when restricted to the plane orthogonal to $u = (u_1, u_2, u_3)$, if u is a unit vector, then U behaves like a rotation by $\pi/2$.

Problem B5 (100 pts). (a) For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map $\exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2,$$

if $\theta \neq 0$, with $\exp(0_3) = I_3$.

(c) Prove that e^A is an orthogonal matrix of determinant $+1$, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \mathbf{SO}(3)$;

- (1) The case $R = I$ is trivial.
- (2) If $R \neq I$ and $\text{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2 \sin \theta} (R - R^T) \mid 1 + 2 \cos \theta = \text{tr}(R) \right\}.$$

(Recall that $\text{tr}(R) = r_{11} + r_{22} + r_{33}$, the *trace* of the matrix R). Note that both θ and $2\pi - \theta$ yield the same matrix $\exp(R)$.)

- (3) If $R \neq I$ and $\text{tr}(R) = -1$, then prove that the eigenvalues of R are $1, -1, -1$, that $R = R^T$, and that $R^2 = I$. Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, S can be diagonalized with respect to an orthogonal matrix Q as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^T.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix}.$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, k \in \mathbb{Z} \right\},$$

where (b, c, d) is any unit vector such that for the corresponding skew symmetric matrix U , we have $U^2 = S$.

(e) To find a skew symmetric matrix U so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get b^2, c^2, d^2 , and then, since one of b, c, d is nonzero, say b , if we choose the positive square root of b^2 , we can determine c and d from bc and bd .

Implement a computer program solve the above system.

Problem B6 (Extra Credit 100 pts). (a) Consider the set of affine maps ρ of \mathbb{R}^3 defined such that

$$\rho(X) = RX + W,$$

where R is a rotation matrix (an orthogonal matrix of determinant +1) and W is some vector in \mathbb{R}^3 . Every such a map can be represented by the 4×4 matrix

$$\begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

Prove that these maps are affine bijections and that they form a group, denoted by **SE**(3) (the *direct affine isometries*, or *rigid motions*, of \mathbb{R}^3).

(b) Let us now consider the set of 4×4 matrices of the form

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix},$$

where Ω is a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and W is a vector in \mathbb{R}^3 .

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^6, +)$. This vector space is denoted by $\mathfrak{se}(3)$. Show that in general, $BC \neq CB$.

(c) Given a matrix

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_3$. Given

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let $\theta = \sqrt{a^2 + b^2 + c^2}$. Prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} I_3 & W \\ 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} e^\Omega & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(d) Prove that

$$e^\Omega = I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2$$

and

$$V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2.$$

Hint. Use the fact that $\Omega^3 = -\theta^2\Omega$.

(e) Prove that e^B is a direct affine isometry in $\mathbf{SE}(3)$. Prove that V is invertible and that

$$Z = I - \frac{1}{2\theta}\Omega + \frac{1}{\theta^2} \left(1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2,$$

for $\theta \neq 0$.

Hint. Assume that the inverse of V is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that a, b , are given by a system of linear equations that always has a unique solution.

Prove that the exponential map $\exp: \mathfrak{se}(3) \rightarrow \mathbf{SE}(3)$ is surjective.

Remark: As in the case of the plane, rigid motions in $\mathbf{SE}(3)$ can be used to describe certain deformations of bodies in \mathbb{R}^3 .

Problem B7 (80 pts). Let A be a real 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

(1) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of A are the roots of the quadratic equation

$$X^2 - \operatorname{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) If we let

$$\mu(A) = \frac{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}{2|a_{11}a_{22} - a_{12}a_{21}|},$$

prove that

$$\operatorname{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) Consider the subset \mathcal{S} of 2×2 matrices whose entries a_{ij} are integers such that $0 \leq a_{ij} \leq 100$.

Prove that the functions $\operatorname{cond}_2(A)$ and $\mu(A)$ reach a maximum on the set \mathcal{S} for the same values of A .

Check that for the matrix

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

we have

$$\mu(A_m) = 19,603 \quad \det(A_m) = -1$$

and

$$\text{cond}_2(A_m) \approx 39,206.$$

(4) Prove that for all $A \in \mathcal{S}$, if $|\det(A)| \geq 2$ then $\mu(A) \leq 10,000$. Conclude that the maximum of $\mu(A)$ on \mathcal{S} is achieved for matrices such that $\det(A) = \pm 1$. Prove that finding matrices that maximize μ on \mathcal{S} is equivalent to finding some integers n_1, n_2, n_3, n_4 such that

$$\begin{aligned} 0 &\leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100 \\ n_1^2 + n_2^2 + n_3^2 + n_4^2 &\geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206 \\ |n_1 n_4 - n_2 n_3| &= 1. \end{aligned}$$

You may use without proof that the fact that the only solution to the above constraints is the multiset

$$\{100, 99, 99, 98\}.$$

(5) Deduce from part (4) that the matrices in \mathcal{S} for which μ has a maximum value are

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \quad \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \quad \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \quad \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}$$

and check that μ has the same value for these matrices. Conclude that

$$\max_{A \in \mathcal{S}} \text{cond}_2(A) = \text{cond}_2(A_m).$$

(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side b by

$$\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \delta b$$

where $y = (y_1, y_2)$. Check that

$$\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\delta x\|_2$, $\|b\|_2$, $\|\delta b\|_2$, and estimate

$$c = \frac{\|\delta x\|_2}{\|x\|_2} \left(\frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Check that

$$c \approx \text{cond}_2(A_m) = 39,206.$$

Problem B8 (20 pts). Let E be a real vector space of finite dimension, $n \geq 1$. Say that two bases, (u_1, \dots, u_n) and (v_1, \dots, v_n) , of E have the *same orientation* iff $\det(P) > 0$, where P the change of basis matrix from (u_1, \dots, u_n) and (v_1, \dots, v_n) , namely, the matrix whose j th columns consist of the coordinates of v_j over the basis (u_1, \dots, u_n) .

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space, E , is the choice of any fixed basis, say (e_1, \dots, e_n) , of E . Any other basis, (v_1, \dots, v_n) , has the *same orientation* as (e_1, \dots, e_n) (and is said to be *positive* or *direct*) iff $\det(P) > 0$, else it is said to have the *opposite orientation* of (e_1, \dots, e_n) (or to be *negative* or *indirect*), where P is the change of basis matrix from (e_1, \dots, e_n) to (v_1, \dots, v_n) . An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \dots, u_n)$ and $B_2 = (v_1, \dots, v_n)$ be two orthonormal bases. For any sequence of vectors, (w_1, \dots, w_n) , in E , let $\det_{B_1}(w_1, \dots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the w_j 's over the basis B_1 and similarly for $\det_{B_2}(w_1, \dots, w_n)$.

Prove that if B_1 and B_2 have the same orientation, then

$$\det_{B_1}(w_1, \dots, w_n) = \det_{B_2}(w_1, \dots, w_n).$$

Given any oriented vector space, E , for any sequence of vectors, (w_1, \dots, w_n) , in E , the common value, $\det_B(w_1, \dots, w_n)$, for all positive orthonormal bases, B , of E is denoted

$$\lambda_E(w_1, \dots, w_n)$$

and called a *volume form* of (w_1, \dots, w_n) .

(c) Given any Euclidean oriented vector space, E , of dimension n for any $n - 1$ vectors, w_1, \dots, w_{n-1} , in E , check that the map

$$x \mapsto \lambda_E(w_1, \dots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted $w_1 \times \dots \times w_{n-1}$, such that

$$\lambda_E(w_1, \dots, w_{n-1}, x) = (w_1 \times \dots \times w_{n-1}) \cdot x,$$

for all $x \in E$. The vector $w_1 \times \dots \times w_{n-1}$ is called the *cross-product* of (w_1, \dots, w_{n-1}) . It is a generalization of the cross-product in \mathbb{R}^3 (when $n = 3$).

Problem B9 (40 pts). Given p vectors (u_1, \dots, u_p) in a Euclidean space E of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors (u_1, \dots, u_p) is the determinant

$$\text{Gram}(u_1, \dots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_p \rangle \\ \langle u_2, u_1 \rangle & \|u_2\|^2 & \dots & \langle u_2, u_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1 \rangle & \langle u_p, u_2 \rangle & \dots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\text{Gram}(u_1, \dots, u_n) = \lambda_E(u_1, \dots, u_n)^2.$$

Hint. If (e_1, \dots, e_n) is an orthonormal basis and A is the matrix of the vectors (u_1, \dots, u_n) over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where A^i denotes the i th column of the matrix A , and $(A^i \cdot A^j)$ denotes the $n \times n$ matrix with entries $A^i \cdot A^j$.

(2) Prove that

$$\|u_1 \times \dots \times u_{n-1}\|^2 = \text{Gram}(u_1, \dots, u_{n-1}).$$

Hint. Letting $w = u_1 \times \dots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \dots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \dots, u_{n-1}, w)^2 = \text{Gram}(u_1, \dots, u_{n-1}, w) \\ &= \text{Gram}(u_1, \dots, u_{n-1})\|w\|^2. \end{aligned}$$

Problem B10 (60 pts). (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

```
function qr( $A$ : matrix): [ $Q, R$ ] pair of matrices
begin
   $n = \text{dim}(A)$ ;
   $R = 0$ ; (the zero matrix)
   $Q1(:, 1) = A(:, 1)$ ;
   $R(1, 1) = \text{sqrt}(Q1(:, 1)^\perp \cdot Q1(:, 1))$ ;
   $Q(:, 1) = Q1(:, 1)/R(1, 1)$ ;
  for  $k := 1$  to  $n - 1$  do
```

```

w = A(:, k + 1);
for i := 1 to k do
    R(i, k + 1) = A(:, k + 1)⊥ · Q(:, i);
    w = w - R(i, k + 1)Q(:, i);
endfor;
Q1(:, k + 1) = w;
R(k + 1, k + 1) = sqrt(Q1(:, k + 1)⊥ · Q1(:, k + 1));
Q(:, k + 1) = Q1(:, k + 1)/R(k + 1, k + 1);
endfor;
end

```

Test it on various matrices, including those involved in Project 1.

(3) Given any invertible matrix A , define the sequences A_k, Q_k, R_k as follows:

$$\begin{aligned}
 A_1 &= A \\
 Q_k R_k &= A_k \\
 A_{k+1} &= R_k Q_k
 \end{aligned}$$

for all $k \geq 1$, where in the second equation, $Q_k R_k$ is the QR decomposition of A_k given by part (2).

Run the above procedure for various values of k (50, 100, ...) and various real matrices A , in particular some symmetric matrices; also run the `Matlab` command `eig` on A_k , and compare the diagonal entries of A_k with the eigenvalues given by `eig(A_k)`.

What do you observe? How do you explain this behavior?

TOTAL: 400 + 260 points + 160 extra.