

**Fundamentals of Linear Algebra and Optimization**  
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**Homework 5 & 6 + Project 3 & 4**

**Note: Problems B2 and B6 are for extra credit**

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**Problem B1 (20 pts).** Let  $A$  be any real or complex  $n \times n$  matrix and let  $\| \cdot \|$  be any operator norm.

Prove that for every  $m \geq 1$ ,

$$\|I\| + \sum_{k=1}^m \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$

Deduce from the above that the sequence  $(E_m)$  of matrices

$$E_m = I + \sum_{k=1}^m \frac{A^k}{k!}$$

converges to a limit denoted  $e^A$ , and called the *exponential* of  $A$ .

**Problem B2 (Extra Credit 60 pts).** Recall that the affine maps  $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where  $\theta, w_1, w_2 \in \mathbb{R}$ , are *rigid motions* (or *direct affine isometries*) and that they form the group **SE**(2).

Given any map  $\rho$  as above, if we let

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then  $\rho$  can be represented by the  $3 \times 3$  matrix

$$A = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & w_1 \\ \sin \theta & \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

(a) Consider the set of matrices of the form

$$\begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$$

where  $\theta, u, v \in \mathbb{R}$ . Verify that this set of matrices is a vector space isomorphic to  $(\mathbb{R}^3, +)$ . This vector space is denoted by  $\mathfrak{se}(2)$ . Show that in general,  $BC \neq CB$ , if  $B, C \in \mathfrak{se}(2)$ .

(b) Given a matrix

$$B = \begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix},$$

prove that if  $\theta = 0$ , then

$$e^B = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

and that if  $\theta \neq 0$ , then

$$e^B = \begin{pmatrix} \cos \theta & -\sin \theta & \frac{u}{\theta} \sin \theta + \frac{v}{\theta} (\cos \theta - 1) \\ \sin \theta & \cos \theta & \frac{u}{\theta} (-\cos \theta + 1) + \frac{v}{\theta} \sin \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

*Hint.* Prove that

$$B^3 = -\theta^2 B,$$

and that

$$e^B = I_3 + \frac{\sin \theta}{\theta} B + \frac{1 - \cos \theta}{\theta^2} B^2.$$

(c) Check that  $e^B$  is a direct affine isometry in  $\mathbf{SE}(2)$ . Prove that the exponential map  $\exp: \mathfrak{se}(2) \rightarrow \mathbf{SE}(2)$  is surjective. If  $\theta \neq k2\pi$  ( $k \in \mathbb{Z}$ ), how do you need to restrict  $\theta$  to get an injective map?

**Problem B3 (100 + 200 pts).** Consider the affine maps  $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where  $\theta, w_1, w_2, \alpha \in \mathbb{R}$ , with  $\alpha > 0$ . These maps are called (direct) *affine similitudes* (for short, *similitudes*). The number  $\alpha > 0$  is the *scale factor* of the similitude. These affine maps are the composition of a rotation of angle  $\theta$ , a rescaling by  $\alpha > 0$ , and a translation.

(a) Prove that these maps form a group that we denote by **SIM**(2).

Given any map  $\rho$  as above, if we let

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then  $\rho$  can be represented by the  $3 \times 3$  matrix

$$A = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha \cos \theta & -\alpha \sin \theta & w_1 \\ \alpha \sin \theta & \alpha \cos \theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha R X + W.$$

(b) Consider the set of matrices of the form

$$\begin{pmatrix} \lambda & -\theta & u \\ \theta & \lambda & v \\ 0 & 0 & 0 \end{pmatrix}$$

where  $\theta, \lambda, u, v \in \mathbb{R}$ . Verify that this set of matrices is a vector space isomorphic to  $(\mathbb{R}^4, +)$ . This vector space is denoted by **sim**(2).

(c) Given a matrix

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

prove that

$$e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

*Hint.* Write

$$\Omega = \lambda I + \theta J,$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe that  $J^2 = -I$ , and prove by induction on  $k$  that

$$\Omega^k = \frac{1}{2} ((\lambda + i\theta)^k + (\lambda - i\theta)^k) I + \frac{1}{2i} ((\lambda + i\theta)^k - (\lambda - i\theta)^k) J.$$

(d) As in (c), write

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

let

$$U = \begin{pmatrix} u \\ v \end{pmatrix},$$

and let

$$B = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}.$$

Prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}U \\ 0 & 0 \end{pmatrix}$$

where  $\Omega^0 = I_2$ .

Prove that

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(e) Prove that

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!} = \int_0^1 e^{\Omega t} dt.$$

Use this formula to prove that if  $\lambda = \theta = 0$ , then

$$V = I_2,$$

else

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix}.$$

Observe that for  $\lambda = 0$ , the above gives back the expression in B2(b) for  $\theta \neq 0$ .

Conclude that if  $\lambda = \theta = 0$ , then

$$e^B = \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix},$$

else

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

with

$$e^\Omega = e^\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

and

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta & -\theta(1 - e^\lambda \cos \theta) - e^\lambda \lambda \sin \theta \\ \theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta & \lambda(e^\lambda \cos \theta - 1) + e^\lambda \theta \sin \theta \end{pmatrix},$$

and that  $e^B \in \mathbf{SIM}(2)$ , with scale factor  $e^\lambda$ .

(f) Prove that the exponential map  $\exp: \mathfrak{sim}(2) \rightarrow \mathbf{SIM}(2)$  is surjective.

(g) Similitudes can be used to describe certain deformations (or flows) of a deformable body  $\mathcal{B}_t$  in the plane. Given some initial shape  $\mathcal{B}$  in the plane (for example, a circle), a deformation of  $\mathcal{B}$  is given by a piecewise differentiable curve

$$\mathcal{D}: [0, T] \rightarrow \mathbf{SIM}(2),$$

where each  $\mathcal{D}(t)$  is a similitude (for some  $T > 0$ ). The deformed body  $\mathcal{B}_t$  at time  $t$  is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}).$$

The surjectivity of the exponential map  $\exp: \mathfrak{sim}(2) \rightarrow \mathbf{SIM}(2)$  implies that there is a map  $\log: \mathbf{SIM}(2) \rightarrow \mathfrak{sim}(2)$ , although it is multivalued. The exponential map and the log “function” allows us to work in the simpler (noncurved) Euclidean space  $\mathfrak{sim}(2)$ .

For instance, given two similitudes  $A_1, A_2 \in \mathbf{SIM}(2)$  specifying the shape of  $\mathcal{B}$  at two different times, we can compute  $\log(A_1)$  and  $\log(A_2)$ , which are just elements of the Euclidean space  $\mathfrak{sim}(2)$ , form the linear interpolant  $(1 - t)\log(A_1) + t\log(A_2)$ , and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1) + t\log(A_2)}, \quad t \in [0, 1].$$

Also, given a sequence of “snapshots” of the deformable body  $\mathcal{B}$ , say  $A_0, A_1, \dots, A_m$ , where each is  $A_i$  is a similitude, we can try to find an interpolating deformation (a curve in  $\mathbf{SIM}(2)$ ) by finding a simpler curve  $t \mapsto C(t)$  in  $\mathfrak{sim}(2)$  (say, a  $B$ -spline) interpolating  $\log A_0, \log A_1, \dots, \log A_m$ . Then, the curve  $t \mapsto e^{C(t)}$  yields a deformation in  $\mathbf{SIM}(2)$  interpolating  $A_0, A_1, \dots, A_m$ .

(1) **(75 pts)**. Write a program interpolating between two deformations.

(2) **(125 pts)**. Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes  $A_0, A_1, \dots, A_m$ .

**Problem B4 (40 pts).** Recall that the coordinates of the cross product  $u \times v$  of two vectors  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  in  $\mathbb{R}^3$  are

$$(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

(a) If we let  $U$  be the matrix

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

check that the coordinates of the cross product  $u \times v$  are given by

$$\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

(b) Show that the set of matrices of the form

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

is a vector space isomorphic to  $(\mathbb{R}^3, +)$ . This vector space is denoted by  $\mathfrak{so}(3)$ . Show that such matrices are never invertible. Find the kernel of the linear map associated with a matrix  $U$ . Describe geometrically the action of the linear map defined by a matrix  $U$ . Show that when restricted to the plane orthogonal to  $u = (u_1, u_2, u_3)$ , if  $u$  is a unit vector, then  $U$  behaves like a rotation by  $\pi/2$ .

**Problem B5 (100 pts).** (a) For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let  $\theta = \sqrt{a^2 + b^2 + c^2}$  and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map  $\exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3)$  is given by

$$\exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2,$$

if  $\theta \neq 0$ , with  $\exp(0_3) = I_3$ .

(c) Prove that  $e^A$  is an orthogonal matrix of determinant  $+1$ , i.e., a rotation matrix.

(d) Prove that the exponential map  $\exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3)$  is surjective. For this, proceed as follows: Pick any rotation matrix  $R \in \mathbf{SO}(3)$ ;

- (1) The case  $R = I$  is trivial.
- (2) If  $R \neq I$  and  $\text{tr}(R) \neq -1$ , then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2 \sin \theta} (R - R^T) \mid 1 + 2 \cos \theta = \text{tr}(R) \right\}.$$

(Recall that  $\text{tr}(R) = r_{11} + r_{22} + r_{33}$ , the *trace* of the matrix  $R$ ). Note that both  $\theta$  and  $2\pi - \theta$  yield the same matrix  $\exp(R)$ .)

- (3) If  $R \neq I$  and  $\text{tr}(R) = -1$ , then prove that the eigenvalues of  $R$  are  $1, -1, -1$ , that  $R = R^T$ , and that  $R^2 = I$ . Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are  $-1, -1, 0$ . Thus,  $S$  can be diagonalized with respect to an orthogonal matrix  $Q$  as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^T.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix}.$$

and use this to conclude that if  $U^2 = S$ , then  $b^2 + c^2 + d^2 = 1$ . Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, k \in \mathbb{Z} \right\},$$

where  $(b, c, d)$  is any unit vector such that for the corresponding skew symmetric matrix  $U$ , we have  $U^2 = S$ .

(e) To find a skew symmetric matrix  $U$  so that  $U^2 = S = \frac{1}{2}(R - I)$  as in (d), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get  $b^2, c^2, d^2$ , and then, since one of  $b, c, d$  is nonzero, say  $b$ , if we choose the positive square root of  $b^2$ , we can determine  $c$  and  $d$  from  $bc$  and  $bd$ .

Implement a computer program solve the above system.

**Problem B6 (Extra Credit 100 pts).** (a) Consider the set of affine maps  $\rho$  of  $\mathbb{R}^3$  defined such that

$$\rho(X) = RX + W,$$

where  $R$  is a rotation matrix (an orthogonal matrix of determinant +1) and  $W$  is some vector in  $\mathbb{R}^3$ . Every such a map can be represented by the  $4 \times 4$  matrix

$$\begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

Prove that these maps are affine bijections and that they form a group, denoted by **SE**(3) (the *direct affine isometries*, or *rigid motions*, of  $\mathbb{R}^3$ ).



(b) Let us now consider the set of  $4 \times 4$  matrices of the form

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix},$$

where  $\Omega$  is a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and  $W$  is a vector in  $\mathbb{R}^3$ .

Verify that this set of matrices is a vector space isomorphic to  $(\mathbb{R}^6, +)$ . This vector space is denoted by  $\mathfrak{se}(3)$ . Show that in general,  $BC \neq CB$ .

(c) Given a matrix

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where  $\Omega^0 = I_3$ . Given

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let  $\theta = \sqrt{a^2 + b^2 + c^2}$ . Prove that if  $\theta = 0$ , then

$$e^B = \begin{pmatrix} I_3 & W \\ 0 & 1 \end{pmatrix},$$

and that if  $\theta \neq 0$ , then

$$e^B = \begin{pmatrix} e^\Omega & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(d) Prove that

$$e^\Omega = I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2$$

and

$$V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2.$$

*Hint.* Use the fact that  $\Omega^3 = -\theta^2\Omega$ .

(e) Prove that  $e^B$  is a direct affine isometry in  $\mathbf{SE}(3)$ . Prove that  $V$  is invertible and that

$$Z = I - \frac{1}{2\theta}\Omega + \frac{1}{\theta^2} \left( 1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2,$$

for  $\theta \neq 0$ .

*Hint.* Assume that the inverse of  $V$  is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that  $a, b$ , are given by a system of linear equations that always has a unique solution.

Prove that the exponential map  $\exp: \mathfrak{se}(3) \rightarrow \mathbf{SE}(3)$  is surjective.

**Remark:** As in the case of the plane, rigid motions in  $\mathbf{SE}(3)$  can be used to describe certain deformations of bodies in  $\mathbb{R}^3$ .

**Problem B7 (80 pts).** Let  $A$  be a real  $2 \times 2$  matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

(1) Prove that the squares of the singular values  $\sigma_1 \geq \sigma_2$  of  $A$  are the roots of the quadratic equation

$$X^2 - \operatorname{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) If we let

$$\mu(A) = \frac{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}{2|a_{11}a_{22} - a_{12}a_{21}|},$$

prove that

$$\operatorname{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) Consider the subset  $\mathcal{S}$  of  $2 \times 2$  matrices whose entries  $a_{ij}$  are integers such that  $0 \leq a_{ij} \leq 100$ .

Prove that the functions  $\operatorname{cond}_2(A)$  and  $\mu(A)$  reach a maximum on the set  $\mathcal{S}$  for the same values of  $A$ .

Check that for the matrix

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

we have

$$\mu(A_m) = 19,603 \quad \det(A_m) = -1$$

and

$$\text{cond}_2(A_m) \approx 39,206.$$

(4) Prove that for all  $A \in \mathcal{S}$ , if  $|\det(A)| \geq 2$  then  $\mu(A) \leq 10,000$ . Conclude that the maximum of  $\mu(A)$  on  $\mathcal{S}$  is achieved for matrices such that  $\det(A) = \pm 1$ . Prove that finding matrices that maximize  $\mu$  on  $\mathcal{S}$  is equivalent to finding some integers  $n_1, n_2, n_3, n_4$  such that

$$\begin{aligned} 0 &\leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100 \\ n_1^2 + n_2^2 + n_3^2 + n_4^2 &\geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206 \\ |n_1 n_4 - n_2 n_3| &= 1. \end{aligned}$$

You may use without proof that the fact that the only solution to the above constraints is the multiset

$$\{100, 99, 99, 98\}.$$

(5) Deduce from part (4) that the matrices in  $\mathcal{S}$  for which  $\mu$  has a maximum value are

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \quad \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \quad \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \quad \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}$$

and check that  $\mu$  has the same value for these matrices. Conclude that

$$\max_{A \in \mathcal{S}} \text{cond}_2(A) = \text{cond}_2(A_m).$$

(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side  $b$  by

$$\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \delta b$$

where  $y = (y_1, y_2)$ . Check that

$$\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute  $\|x\|_2$ ,  $\|\delta x\|_2$ ,  $\|b\|_2$ ,  $\|\delta b\|_2$ , and estimate

$$c = \frac{\|\delta x\|_2}{\|x\|_2} \left( \frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Check that

$$c \approx \text{cond}_2(A_m) = 39,206.$$

**Problem B8 (20 pts).** Let  $E$  be a real vector space of finite dimension,  $n \geq 1$ . Say that two bases,  $(u_1, \dots, u_n)$  and  $(v_1, \dots, v_n)$ , of  $E$  have the *same orientation* iff  $\det(P) > 0$ , where  $P$  the change of basis matrix from  $(u_1, \dots, u_n)$  and  $(v_1, \dots, v_n)$ , namely, the matrix whose  $j$ th columns consist of the coordinates of  $v_j$  over the basis  $(u_1, \dots, u_n)$ .

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space,  $E$ , is the choice of any fixed basis, say  $(e_1, \dots, e_n)$ , of  $E$ . Any other basis,  $(v_1, \dots, v_n)$ , has the *same orientation* as  $(e_1, \dots, e_n)$  (and is said to be *positive* or *direct*) iff  $\det(P) > 0$ , else it is said to have the *opposite orientation* of  $(e_1, \dots, e_n)$  (or to be *negative* or *indirect*), where  $P$  is the change of basis matrix from  $(e_1, \dots, e_n)$  to  $(v_1, \dots, v_n)$ . An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let  $B_1 = (u_1, \dots, u_n)$  and  $B_2 = (v_1, \dots, v_n)$  be two orthonormal bases. For any sequence of vectors,  $(w_1, \dots, w_n)$ , in  $E$ , let  $\det_{B_1}(w_1, \dots, w_n)$  be the determinant of the matrix whose columns are the coordinates of the  $w_j$ 's over the basis  $B_1$  and similarly for  $\det_{B_2}(w_1, \dots, w_n)$ .

Prove that if  $B_1$  and  $B_2$  have the same orientation, then

$$\det_{B_1}(w_1, \dots, w_n) = \det_{B_2}(w_1, \dots, w_n).$$

Given any oriented vector space,  $E$ , for any sequence of vectors,  $(w_1, \dots, w_n)$ , in  $E$ , the common value,  $\det_B(w_1, \dots, w_n)$ , for all positive orthonormal bases,  $B$ , of  $E$  is denoted

$$\lambda_E(w_1, \dots, w_n)$$

and called a *volume form* of  $(w_1, \dots, w_n)$ .

(c) Given any Euclidean oriented vector space,  $E$ , of dimension  $n$  for any  $n - 1$  vectors,  $w_1, \dots, w_{n-1}$ , in  $E$ , check that the map

$$x \mapsto \lambda_E(w_1, \dots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted  $w_1 \times \dots \times w_{n-1}$ , such that

$$\lambda_E(w_1, \dots, w_{n-1}, x) = (w_1 \times \dots \times w_{n-1}) \cdot x,$$

for all  $x \in E$ . The vector  $w_1 \times \dots \times w_{n-1}$  is called the *cross-product* of  $(w_1, \dots, w_{n-1})$ . It is a generalization of the cross-product in  $\mathbb{R}^3$  (when  $n = 3$ ).

**Problem B9 (40 pts).** Given  $p$  vectors  $(u_1, \dots, u_p)$  in a Euclidean space  $E$  of dimension  $n \geq p$ , the *Gram determinant* (or *Gramian*) of the vectors  $(u_1, \dots, u_p)$  is the determinant

$$\text{Gram}(u_1, \dots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_p \rangle \\ \langle u_2, u_1 \rangle & \|u_2\|^2 & \dots & \langle u_2, u_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1 \rangle & \langle u_p, u_2 \rangle & \dots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\text{Gram}(u_1, \dots, u_n) = \lambda_E(u_1, \dots, u_n)^2.$$

*Hint.* If  $(e_1, \dots, e_n)$  is an orthonormal basis and  $A$  is the matrix of the vectors  $(u_1, \dots, u_n)$  over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where  $A^i$  denotes the  $i$ th column of the matrix  $A$ , and  $(A^i \cdot A^j)$  denotes the  $n \times n$  matrix with entries  $A^i \cdot A^j$ .

(2) Prove that

$$\|u_1 \times \dots \times u_{n-1}\|^2 = \text{Gram}(u_1, \dots, u_{n-1}).$$

*Hint.* Letting  $w = u_1 \times \dots \times u_{n-1}$ , observe that

$$\lambda_E(u_1, \dots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \dots, u_{n-1}, w)^2 = \text{Gram}(u_1, \dots, u_{n-1}, w) \\ &= \text{Gram}(u_1, \dots, u_{n-1})\|w\|^2. \end{aligned}$$

**Problem B10 (60 pts).** (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

```

function qr( $A$ : matrix): [ $Q, R$ ] pair of matrices
  begin
     $n = \text{dim}(A)$ ;
     $R = 0$ ; (the zero matrix)
     $Q1(:, 1) = A(:, 1)$ ;
     $R(1, 1) = \text{sqrt}(Q1(:, 1)^\perp \cdot Q1(:, 1))$ ;
     $Q(:, 1) = Q1(:, 1)/R(1, 1)$ ;
    for  $k := 1$  to  $n - 1$  do

```

```

w = A(:, k + 1);
for i := 1 to k do
    R(i, k + 1) = A(:, k + 1)⊥ · Q(:, i);
    w = w - R(i, k + 1)Q(:, i);
endfor;
Q1(:, k + 1) = w;
R(k + 1, k + 1) = sqrt(Q1(:, k + 1)⊥ · Q1(:, k + 1));
Q(:, k + 1) = Q1(:, k + 1)/R(k + 1, k + 1);
endfor;
end

```

Test it on various matrices, including those involved in Project 1.

(3) Given any invertible matrix  $A$ , define the sequences  $A_k, Q_k, R_k$  as follows:

$$\begin{aligned}
 A_1 &= A \\
 Q_k R_k &= A_k \\
 A_{k+1} &= R_k Q_k
 \end{aligned}$$

for all  $k \geq 1$ , where in the second equation,  $Q_k R_k$  is the QR decomposition of  $A_k$  given by part (2).

Run the above procedure for various values of  $k$  (50, 100, ...) and various real matrices  $A$ , in particular some symmetric matrices; also run the `Matlab` command `eig` on  $A_k$ , and compare the diagonal entries of  $A_k$  with the eigenvalues given by `eig`( $A_k$ ).

What do you observe? How do you explain this behavior?

**TOTAL: 400 + 260 points + 160 extra.**